C4 June 2009

(1) $f(x) = \dfrac{1}{\sqrt{4+x}} = (4+x)^{-\frac{1}{2}}$

$= \left[4\left(1+\frac{x}{4}\right)\right]^{-\frac{1}{2}} = \frac{1}{2}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$

$= \frac{1}{2}\left(1 + \left(-\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{4}\right)^3}{3!} + \cdots\right)$

$= \frac{1}{2}\left(1 - \frac{x}{8} + \frac{3}{128}x^2 - \frac{5}{1024}x^3 + \cdots\right)$

$= \underline{\underline{\frac{1}{2} - \frac{x}{16} + \frac{3}{256}x^2 - \frac{5}{2048}x^3 + \cdots}}$

(2) $y = 3\cos\left(\frac{x}{3}\right), \quad 0 \le x \le \frac{3\pi}{2}$

(a)

x	0	$3\pi/8$	$3\pi/4$	$9\pi/8$	$\frac{3\pi}{2}$
y	3	2.77164	2.12132	$\boxed{1.14805}$	0

(b) Area of $R \approx \dfrac{\left(\frac{3\pi}{8}\right)}{2}\left(3+0+2(2.77164+2.12132+1.14805)\right)$

$= \underline{\underline{8.884}}$

(c) Exact area of $R = \int_0^{\frac{3\pi}{2}} 3\cos\left(\frac{x}{3}\right)dx = \left[9\sin\left(\frac{x}{3}\right)\right]_0^{3\pi/2}$

$= \underline{\underline{9}}$

③ (a) $f(x) = \dfrac{4-2x}{(2x+1)(x+1)(x+3)} = \dfrac{A}{2x+1} + \dfrac{B}{x+1} + \dfrac{C}{x+3}$

$\Rightarrow \; 4-2x = A(x+1)(x+3) + B(2x+1)(x+3) + C(2x+1)(x+1).$

$x=-1 \Rightarrow \; 6 = -2B \Rightarrow \underline{B=-3}$

$x=-3 \Rightarrow \; 10 = 10C \Rightarrow \underline{C=1}$

$x=-\tfrac{1}{2} \Rightarrow \; 5 = \tfrac{5}{4}A \Rightarrow \underline{A=4}$

(b) (i) $\displaystyle\int f(x)\,dx = \int \frac{4}{2x+1} - \frac{3}{x+1} + \frac{1}{x+3}\,dx$

$= 4\,\dfrac{\ln|2x+1|}{2} - 3\ln|x+1| + \ln|x+3| + C.$

$= \underline{2\ln|2x+1| - 3\ln|x+1| + \ln|x+3| + C}$

(ii) $\displaystyle\int_0^2 f(x)\,dx = (2\ln 5 - 3\ln 3 + \ln 5) - (\ln 3)$

$= 3\ln 5 - 4\ln 3 = \underline{\underline{\ln \tfrac{125}{81}}}$

4 (a) $ye^{-2x} = 2x + y^2$

Differentiate both sides w.r.t. x

$\Rightarrow \quad y(-2e^{-2x}) + \frac{dy}{dx} \cdot e^{-2x} = 2 + 2y\frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx}(e^{-2x} - 2y) = 2 + 2ye^{-2x}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2 + 2ye^{-2x}}{e^{-2x} - 2y}$

(b) $P(0, 1)$.

At P, $\frac{dy}{dx} = \frac{2+2}{1-2} = -4$

$\Rightarrow$ gradient of normal is $\frac{1}{4}$.

Equation of normal is $y - 1 = \frac{1}{4}(x - 0)$

$\Rightarrow \quad x - 4y + 4 = 0$

5 (a) $x = 2\cos 2t, \ y = 6\sin t, \quad 0 \le t \le \frac{\pi}{2}$.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6\cos t}{-4\sin 2t} = \frac{-3\cos t}{2\sin 2t}$

when $t = \pi/3$, $\frac{dy}{dx} = \frac{-3\cos\frac{\pi}{3}}{2\sin 2\pi/3} = -\frac{\sqrt{3}}{2}$

(b) $x = 2\cos 2t = 2(1 - 2\sin^2 t) = 2 - 4\sin^2 t$

$= 2 - 4\left(\frac{y}{6}\right)^2 = 2 - \frac{y^2}{9} \Rightarrow y^2 = 18 - 9x$

From sketch we want +ve square root,

so $y = \sqrt{18 - 9x}, \ -2 \le x \le 2$

N.B. $0 \le t \le \frac{\pi}{2}$

$\Rightarrow 0 \le 2t \le \pi$

$\Rightarrow -1 \le \cos 2t \le 1 \Rightarrow -2 \le 2\cos 2t \le 2$.

(c) Range is $\underline{0 \le f(x) \le 6}$

since maximum value is $\sqrt{18-9(-2)} = 6$.

6 (a) $$\int \sqrt{5-x}\, dx = \int (5-x)^{\frac{1}{2}}\, dx$$

$$= \frac{(5-x)^{3/2}}{(\frac{3}{2})(-1)} + C = \underline{-\frac{2}{3}(5-x)^{\frac{3}{2}} + C}$$

(b) (i) $\int (x-1)\sqrt{5-x}\, dx$ Using integration by parts,

$$= (x-1)\left(-\frac{2}{3}(5-x)^{\frac{3}{2}}\right) - \int 1 \cdot \left(-\frac{2}{3}(5-x)^{\frac{3}{2}}\right) dx$$

$$= -\frac{2}{3}(x-1)(5-x)^{3/2} + \frac{2}{3}\int (5-x)^{3/2}\, dx$$

$$= \underline{-\frac{2}{3}(x-1)(5-x)^{\frac{3}{2}} - \frac{4}{15}(5-x)^{5/2} + C}$$

Method 2 $\int (x-1)\sqrt{5-x}\, dx$ Put $u^2 = 5-x$

$2u\frac{du}{dx} = -1$

$\underline{dx = -2u\,du}$

$x-1 = 4-(5-x) = 4-u^2$

$$= \int (4-u^2)(u)(-2u)\, du$$

$$= \int 2u^4 - 8u^2\, du$$

$$= \frac{2u^5}{5} - \frac{8u^3}{3} + C$$

$$= \underline{\frac{2}{5}(5-x)^{5/2} - \frac{8}{3}(5-x)^{3/2} + C}$$

(ii) $\displaystyle\int_1^5 (x-1)\sqrt{5-x}\,dx$

$$= \left[-\frac{2}{3}(x-1)(5-x)^{3/2} - \frac{4}{15}(5-x)^{5/2}\right]_1^5$$

$$= 0 - \left(0 - \frac{4}{15}\left(4^{5/2}\right)\right) = \frac{4}{15} \times 2^5 = \underline{\underline{\frac{128}{15}}}$$

Method 2

Using $u^2 = 5-x$, $x=1 \Rightarrow u=2$
$x=5 \Rightarrow u=0$

$$\longrightarrow \left[\frac{2}{5}u^5 - \frac{8}{3}u^3\right]_2^0 = 0 - \left(\frac{2}{5}\times 2^5 - \frac{8}{3}\times 2^3\right)$$

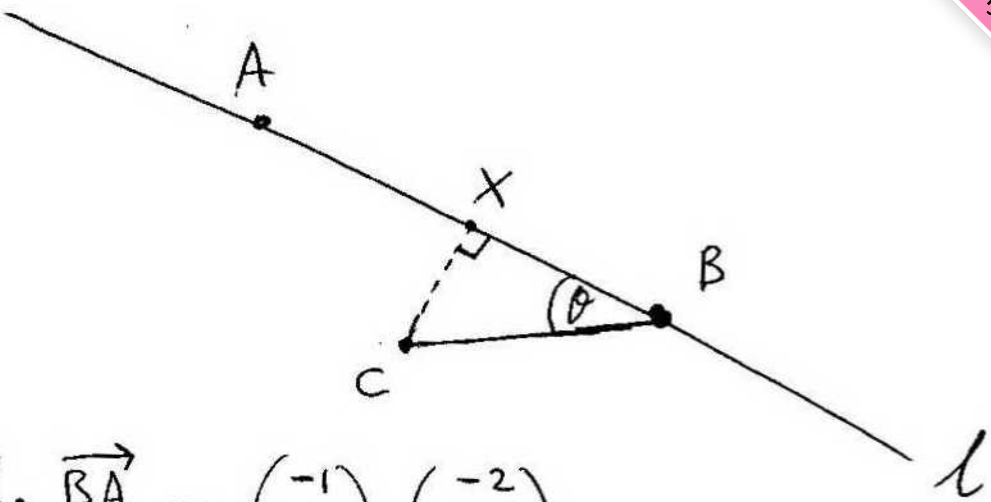
$$= \frac{64}{3} - \frac{64}{5} = \underline{\underline{\frac{128}{15}}}$$

7 (a) Equation of l : $\underline{r} = \begin{pmatrix} 8 \\ 13 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

(b) $|\overrightarrow{CB}| = |\underline{b} - \underline{c}| = \left|\begin{pmatrix} 1 \\ 5 \\ -10 \end{pmatrix}\right| = \sqrt{1^2+5^2+(-10)^2}$

$= \sqrt{126} = \underline{3\sqrt{14}}$

(c)



$$\cos\theta = \frac{\overrightarrow{BC} \cdot \overrightarrow{BA}}{|\overrightarrow{BC}|\ |\overrightarrow{BA}|} = \frac{\begin{pmatrix}-1\\-5\\10\end{pmatrix} \cdot \begin{pmatrix}-2\\-1\\2\end{pmatrix}}{3\sqrt{14} \times 3}$$

$$= \frac{27}{9\sqrt{14}} \Rightarrow \underline{\theta = 36.7^\circ}.$$

(d) CX is shortest distance from C to the line,

where $\frac{CX}{CB} = \sin\theta \Rightarrow CX = 3\sqrt{14} \times \sin\theta$.

$$= \underline{6.708}$$

(e) Area of $\triangle CXB = \frac{1}{2} \cdot CX \cdot CB \cdot \sin(90-\theta)$

$$= \frac{1}{2} \times (6.708\ldots) \times 3\sqrt{14} \times \sin(90-\theta)$$

$$= \underline{30.2 \text{ units}^2}$$

8 (a) $\int \sin^2\theta \, d\theta = \int \frac{1}{2}(1 - \cos 2\theta)\, d\theta$

$= \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + c = \underline{\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + c}$

(b) $x = \tan\theta, \quad y = 2\sin 2\theta, \quad 0 \le \theta < \frac{\pi}{2}$

$V = \int_0^{1/\sqrt{3}} \pi y^2 \, dx$

$x = 0 \Rightarrow \theta = 0$

$x = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$

So $V = \int_0^{\pi/6} \pi y^2 \frac{dx}{d\theta} \, d\theta$

$= \int_0^{\pi/6} \pi (4\sin^2 2\theta) \sec^2\theta \, d\theta$

$= \int_0^{\pi/6} \pi \left(4 (2\sin\theta\cos\theta)^2\right) \times \frac{1}{\cos^2\theta} \, d\theta$

$= \int_0^{\pi/6} 16\pi \sin^2\theta \, d\theta$ so $k = 16\pi$.

(c) $= \left[16\pi\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\right]_0^{\pi/6}$

$= 16\pi\left(\left(\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right) - 0\right) = \underline{\underline{\frac{4}{3}\pi^2 - 2\pi\sqrt{3}}}$